

Quotients by finite group actions

Wednesday, April 5, 2023 11:11 PM

Main goals for today:

1) Finite affine GIT quotients are geometric

2) DM stacks are locally quotient stacks.

§1

Recall:

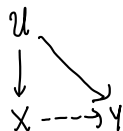
e.g. G -equivariant for
trivial G -action on X .

Def: If finite gp $G \curvearrowright U$ alg space,

a G -invariant morph $U \rightarrow X$ is
a geometric quotient if

1) $U(\bar{k})/G \xrightarrow{\sim} X(\bar{k}) \quad \forall \bar{k} \text{ alg closed}$

2) The morphism $U \rightarrow X$ is initial
for G -invariant morphisms to alg. spaces:



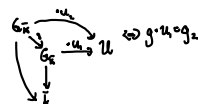
Prop: If $G \curvearrowright U$ freely, so that the

$[U/G]$ is an algebraic space, then

$U \rightarrow [U/G]$ is a geom. quotient.

Pf: 1) follows from fact that a principal G -bundle
over $\bar{k}$ is just $G_{\bar{k}} \curvearrowright u \in U(\bar{k})$ maps to $G_{\bar{k}} \xrightarrow{u} U$

u_1, u_2 map to same orbit $\iff$ from



2) can be checked on level of presheaves/prestacks

(G still finite)

Let's consider the case $G \curvearrowright \text{Spec } A$.

$\rightsquigarrow G \curvearrowright A$, and

we can consider

$\text{Spec } A \rightarrow \text{Spec } A^G$

(affine GIT quotient)

Ex: $G = \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathrm{SL}_2 \mathbb{C} \hookrightarrow \mathbb{A}_{\mathbb{C}}^2$

$$k \longmapsto \begin{bmatrix} e^{\frac{2\pi i k}{n}} & \\ & e^{-\frac{2\pi i k}{n}} \end{bmatrix}$$

$$\leadsto G \hookrightarrow \mathbb{C}[x, y]$$

$$k \cdot x = e^{\frac{2\pi i k}{n}} x$$

$$k \cdot y = e^{-\frac{2\pi i k}{n}} y$$

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^n, xy, y^n]$$

Lemma: G finite $\hookrightarrow \mathrm{Spec} A$

$$\leadsto \mathrm{Spec} A \longrightarrow \mathrm{Spec} A^G \text{ integral.}$$

If A is a f.g. $\mathbb{R}$ -alg, $\mathbb{R}$ noether

$$\leadsto \mathrm{Spec} A^G \text{ finite / } \mathrm{Spec} A \text{ and}$$

$$A^G \text{ f.g. } \mathbb{R}\text{-alg.}$$

Pf: omitted.

Lemma: G finite $\hookrightarrow \mathrm{Spec} A$.

If $A^G \longrightarrow B$ flat, then

$$G \hookrightarrow \mathrm{Spec}(A \otimes B) =: \mathrm{Spec} C$$

$$\text{and } B = C^G.$$

Pf: Apply tensor prod w/ B to

$$0 \rightarrow A^G \rightarrow A \xrightarrow[p_2]{p_1} \prod_{g \in G} A$$

$$p_1(f) = (f)_{g \in G}$$

$$p_2(f) = (gf)_{g \in G}.$$



Thm: G finite $\hookrightarrow \mathrm{Spec} A$.

$$\leadsto \mathrm{Spec} A \longrightarrow \mathrm{Spec} A^G \text{ is a geometric quotient.}$$

$$0 \rightarrow \dots \rightarrow \prod_{g \in G} A \xrightarrow{\pi} A^G \rightarrow 0$$

PF: Let $U = \text{Spec } A \longrightarrow \text{Spec } A$

Two things to check:

1) On $\bar{k}$ -points, $U(\bar{k}) \longrightarrow X(\bar{k})$ is quotient by G .

i.e. π induces a bijection between G -orbits of $\bar{k}$ -points of U and $\bar{k}$ -points of X .

Clearly $U(\bar{k}) \longrightarrow X(\bar{k})$ is surjective

since π is integral ($\Rightarrow$ closed) & dominant.

Suppose $u_1, u_2 \in U(\bar{k}) \longmapsto x \in X(\bar{k})$.

$$\begin{array}{ccc} \text{Consider } \text{Spec } A \otimes_{\bar{k}} \bar{k} = U_u & \longrightarrow & x \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

$G \cdot u_1, G \cdot u_2 \subseteq U_u$ are closed

(finite unions of $\bar{k}$ -points)

If they are disjoint, then since U_u is

affine $\Rightarrow$ separated, $\exists f \in A$ s.t. on

the fiber $U_u = \text{Spec}(A \otimes_{\bar{k}} \bar{k})$, $f|_{U_u} = 0$ on $G \cdot u_1$, $f \neq 0$ on $G \cdot u_2$.

"Averaging" f over G :

$$\begin{array}{l} \text{Take } f' = \pi(gf) \stackrel{?}{=} 0 \text{ on } G \cdot u_1 \\ \uparrow \\ A^G \end{array}$$

contradicting that $u_1, u_2 \longmapsto x \in \text{Spec } A^G$.

$$\Rightarrow U(\bar{k})/G \xrightarrow{\sim} X(\bar{k}).$$

2) Want to show universal property for G -invariant maps, i.e.

$$\begin{array}{ccc} U & & \\ \pi \downarrow & \searrow & \swarrow \\ X & \dashrightarrow & Y \end{array} \quad \text{any algebraic space.}$$

$\Updownarrow$

In the diagram

$$\begin{array}{ccc} U & & \\ \downarrow & \searrow \pi & \\ [U/G] & \xrightarrow{\overline{\pi}} & X \end{array}$$

Want to the natural map

$$\text{Hom}(X, Y) \longrightarrow \text{Hom}([U/G], Y)$$

is an isom. Y algebraic space.

Inj:

Fix $h_1, h_2: X \rightarrow Y$ s.t. $h_1 \circ \overline{\pi} = h_2 \circ \overline{\pi}$.

Then consider

$$\begin{array}{ccccc} U & & \xrightarrow{\pi} & & \\ \downarrow & & \searrow \overline{\pi} & & \\ [U/G] & & \xrightarrow{\overline{\pi}} & & X \\ & \searrow \mathcal{U} & & \searrow (h_1, h_2) & \\ & E & \xrightarrow{\mathcal{U}} & & Y \times Y \\ & \downarrow \Delta & & & \\ & Y & \xrightarrow{\Delta} & & Y \times Y \end{array}$$

$h_1 \circ \overline{\pi}$

Y alg space $\Rightarrow \Delta$ loc. of f.t., mono

$\Rightarrow \mathcal{U}$ loc. of f.t., mono.

OTOH, π is universally closed (integral)

$\Rightarrow \mathcal{U}$ is universally closed mono

loc. of f.t. $\rightarrow \mathcal{U}$ closed immersion

OTOH,

$$\mathcal{O}_x \rightarrow \pi_* \mathcal{O}_U \text{ inj}$$

$$\Rightarrow \mathcal{O}_x \rightarrow \mathcal{U}_* \mathcal{O}_E \text{ inj}$$

$\Rightarrow \mathcal{U}$ is an isom.

(Most delicate part)

Surj: Take any $g: X \rightarrow Y$, Y alg space.

Want a factorization.

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow & \nearrow \\ & & Y \end{array}$$

^

①: Since X is quasi-affine we may assume Y is by replacing Y by quasi-affine open subg subspace of Y .

$\leadsto Y$ admits étale cover by an affine scheme.

②: Existence of factorization can be checked ét. loc. on X :

let $V \rightarrow X$ étale cover s.t. $\exists$ a factorization

$$\begin{array}{ccc} V \times_X X & \rightarrow & X \xrightarrow{g} Y \\ & \searrow & \nearrow h \\ & V & \end{array}$$

Consider the diagram

$$\begin{array}{ccccc} V \times_X V \times_X X & \rightrightarrows & V \times_X X & \rightarrow & X \\ \downarrow & \square & \downarrow & \square & \downarrow \\ V \times_X V \times_X X & \rightrightarrows & V & \rightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V \times_X V & & & & Y \end{array}$$

Note that a similar argument to above using coequalizers gives that $V \times_X V \rightrightarrows Y$ agree (since $V \times_X V \times_X X \rightrightarrows Y$ agree)

$\leadsto$ étale descent gives the required map $X \rightarrow Y$.

③: Write $Y' \rightarrow Y$ étale cover, Y' affine.

Let $X' = X \times_Y Y'$.

By ② it suffices to find $V \xrightarrow{\text{ét}} X, V$ affine s.t.

$$\begin{array}{ccccc} & X_{Y'} & & & \\ & \downarrow & & & \\ X_{Y'} & \xrightarrow{s} & X_{Y'} & \rightarrow & V \\ \downarrow & \square & \downarrow & \square & \downarrow \\ X' & \rightarrow & X & \rightarrow & X \end{array}$$

$$\begin{array}{ccc} \downarrow & \dashrightarrow & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

(étale-local section to $X' \rightarrow X$)

Claim: If s exists as above, then the desired map

$V \rightarrow X$ exists too:

$$\Gamma(Y', \mathcal{O}_{Y'}) \rightarrow \Gamma(X_V, \mathcal{O}_{X_V}) = \Gamma(V, \mathcal{O}_V)$$

Y' and V affine

$\leadsto$ get desired map $V \rightarrow Y' \rightarrow Y$.

Later steps are somewhat technical...

④: Reduce to $X = \text{Spec } A^G$ strictly henselian.

By construction we had $U = \text{Spec } A_i$, $X = \text{Spec } A^G$,

$X_0 = [\text{Spec } A_i / G]$.

$$\begin{array}{ccc} u & U & \pi \\ \downarrow & \downarrow & \searrow \\ X_0 & \xrightarrow{\pi} & X \\ \pi \uparrow & & \uparrow \pi(x) \end{array}$$

Now $X^{\text{sh}} = \text{Spec } \mathcal{O}_{X, \pi(x)}^{\text{sh}}$ is local ring in étale top
 $\Rightarrow X^{\text{sh}} = \varprojlim X_i$, X_i affine ét. nbhd of $\pi(x) \in X$.

$$\begin{array}{ccccc} & & U^{\text{sh}} & & \\ & \swarrow & \downarrow & \searrow & \\ \leadsto & U_i & \rightarrow & X^{\text{sh}} & \\ & \downarrow & \downarrow & \downarrow & \\ U & \rightarrow & X_i & \rightarrow & X \\ & \downarrow & \downarrow & \downarrow & \\ X & \rightarrow & X & \rightarrow & X \end{array}$$

From step ③: étale loc. over X , wanted a section
 $X \rightarrow X'$.

If we can construct a map $X^{\text{sh}} \rightarrow X'$,
 then since

$$\text{colim}_i \text{Mor}_{X_i}(Y_i, X') \xrightarrow{\sim} \text{Mor}_{X_0}(\varprojlim_i X_i, X') \quad \star$$

we therefore have an étale-local section $X_i \rightarrow X'$.

Since invariants commutes w/ flat basechange,

$U^{\text{sh}} \rightarrow X^{\text{sh}}$ is the map of affine schemes given by
 taking G -invariants.

may assume X strictly henselian.

★: Not obvious!

This is opposite from commutative red' get via tensor-hom adjunction, & relies on the following:

Ex 3.32) fact: If $f: X \rightarrow Y$ map of alg stacks

loc. of finite pres. then

$$\text{colim}_{\mathcal{U}} \text{Mor}_{\mathcal{Y}}(\text{Spec } A_i, X) \xrightarrow{\sim} \text{Mor}_{\mathcal{Y}}(\text{Spec colim } A_i, X)$$

for any directed system of algebras over $\mathcal{Y}$.

⑤ If X strictly henselian. (oh) (See section A.9)

$$\begin{array}{ccc} U & \longrightarrow & X_i \text{ finite} \\ \uparrow & & \uparrow \\ \text{Spec } A & & \text{Spec } A^G \end{array}$$

then $A = \prod A_i$ product of sh local rings.

let $u_i \in \text{Spec } A_i$ closed pt.

Then $G_{u_i} = \text{stab}(u_i) \leq G$ fixes $\text{Spec } A_i$,

& $G \cdot u_i = \{\text{closed pts of } U_i\}$

$$\leadsto X_i = [\text{Spec } A_i / G_{u_i}]$$

(this trick is called "slicing groupoids", see 3.4.2).

Now given $Y' \rightarrow Y$ as above, since $Y' \rightarrow Y$ representable by schemes, can write $X' = [U'_i / G_{u_i}]$, where

$$\begin{array}{ccc} U'_i & \xrightarrow{\sigma} & \text{Spec } A_i =: U_i \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\sigma} & Y \end{array}$$

$\leadsto \exists U_i \rightarrow U'_i$ section
(Prop A.9.3)

Check G_{u_i} invariance w.r.t. sections to section $U_i \rightarrow X'$.



Cor: finite gp $G \curvearrowright \text{Spec } A$ freely

$\leadsto [\text{Spec } A / G] = \text{Spec } A^G$
is in fact an affine scheme.

Prin: Thm above generalizes to finite flat groupoids of affine schemes. (Ex 4.2.10)

§2: Local structure of DM stacks

Goal: Every DM stack is étale locally a quotient by the stabilizers of points.

More precisely:

Defn: If $x \in X$ DM stack, the geometric stabilizer G_x is the discrete group given by the stabilizer of a geometric point $\text{Spec } \bar{k} \rightarrow X$ with image x .

Note: There is a construction which is useful in the following proof: there is a natural equivalence

$$\left\{ \begin{array}{c} \text{degree } d \text{ étale covers} \\ T \rightarrow S \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} S_{d+1}\text{-torsors} \\ P \rightarrow S \end{array} \right\}$$

$$(T \rightarrow S) \longmapsto (\overbrace{T \times_S \dots \times_S T}^{d \text{ times}} \setminus \Delta \rightarrow S)$$

↑ union of pairwise diagonals

$$(P/S_{d+1} \rightarrow S) \longleftrightarrow (P \rightarrow S)$$

where $S_{d+1} = S_d$ by forgetting the last entry.

Thm X a sep. DM stack, $x \in X$ w/ geom stab G_x . For an affine étale map

$$f: ([\text{Spec } A/G_x], w) \rightarrow (X, x)$$

s.t. f induces an isom on geom. stabilizers at $w \in [\text{Spec } A/G_x]$.

pf: Let $\text{Spec } \bar{k} \rightarrow X$ represent x , geom pt.

Then X admits a representable étale map from a scheme

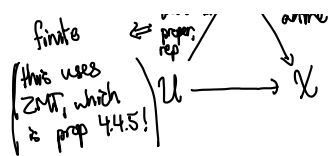
$$(U, w) \xrightarrow{j} (X, x)$$

and shrinking U , we may assume U affine, and j quasi-finite.

Now since X is sep, the diagonal is proper $\rightsquigarrow$

$$U \times X$$

$$\text{quasi-finite} \rightarrow \text{closed}$$



$$\leadsto U \longrightarrow X \text{ affine.}$$

let $d = |Z|$, where

$$\begin{array}{ccc} Z & \longrightarrow & \text{Spec } k \\ \downarrow \square & & \downarrow \\ U & \longrightarrow & X \end{array}$$

Now let $(U/X)^d := \overbrace{U \times_X \cdots \times_X U}^{d \text{ times}};$
 this an aff. sch & $\exists$ bij.

$$\left\{ \text{maps } S \rightarrow (U/X)^d \right\} \leftrightarrow \left\{ \text{maps } S \rightarrow X \text{ w/ sections } s_1, \dots, s_d: S \rightarrow U_{s_i} \text{ of } \pi: U_{s_i} \rightarrow S \right\}$$

We can define $(U/X)_o^d = (U/X)^d$
 as the open subscheme given by the complement of
 all pairwise divergences.

$\rightarrow$ pairwise disjoint

$$\left\{ \text{maps } S \rightarrow (U/X)_o^d \right\} \leftrightarrow \left\{ \text{maps } S \rightarrow X \text{ w/ sections } s_1, \dots, s_d: S \rightarrow U_{s_i} \text{ of } \pi: U_{s_i} \rightarrow S \right\},$$

and $(U/X)_o^d$ quasi-affine.
 $S_d \curvearrowright (U/X)_o^d$ by permuting sections

Given

$$\begin{array}{ccc} Z & \longrightarrow & U \\ \text{finite} \downarrow \square & & \downarrow \\ \text{state} & & \text{Spec } k \longrightarrow X \end{array}$$

can do the same construction

$$\leadsto \begin{array}{ccc} S_d \curvearrowright (Z/\text{Spec } k)_o^d & \longrightarrow & (U/X)_o^d \\ \downarrow S_d \text{-tors} & & \downarrow \\ \text{Spec } k & \longrightarrow & X \end{array}$$

(by defn of fiber prod!)

Now since $G_m \subset Z$ naturally we get an
 inclusion $G_m \subset S_d \Rightarrow G_m$ acts on $(U/X)_o^d$.

$$\leadsto \begin{array}{ccc} [(U/X)_o^d / G_m] & & \downarrow \\ & & X \end{array}$$

which satisfies the stabilizer axiom
at the given pt w corr to

$$\begin{array}{ccc} G_m \subset S_d \cong (Z/\text{Spec } k)^d & \longrightarrow & (U/X)_0^d \\ \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & X \end{array}$$

let $W = (U/X)_0^d$ be a G_k -invariant affine open
containing this copy of G_m over $\text{Spec } k$;
then $f: [W/G_m] \rightarrow [(U/X)_0^d/G_m] \rightarrow X$
is the desired map.

$$\begin{array}{l} U \text{ étale, representable,} \\ \downarrow \text{ affine} \\ X \\ \Rightarrow W \text{ étale, representable, affine} \\ \downarrow \\ X \end{array}$$

Want that $[W/G_m] \rightarrow X$ ét. rep. affine.

let V be any scheme.

$$\begin{array}{ccccc} W \times_X V & \xrightarrow{\text{ét}} & [W/G_m] \times_X V & \longrightarrow & V \\ \downarrow \square & & \downarrow \square & & \downarrow \\ W & \xrightarrow{\text{ét}} & [W/G_m] & \longrightarrow & X \end{array}$$

ét

$\Rightarrow$ étale $\checkmark$

$$W \times_X V = (U/X)_0^d \times_X V = (U \times_X V / V)_0^d$$

$\uparrow$
alg space w/ free G_m -action
(since free S_d -action)

$$\Rightarrow [W/G_m] \times_X V = [W \times_X V / G_m]$$

alg space.

By cor from before, if $V = \text{aff. sch.}$,

$W \times_x V$ is aff. sch

$\Rightarrow$ by cor to before,
finite gp quotient by free action is
affine

$$\Rightarrow [W/G_x] \rightarrow X$$

ét, rep, affine.



Same strategy of turning finite étale covers
into S_x -torsors can be used to prove more
results, e.g.:

Fact: For X DM stacks,

$$X \cong [U/G] \quad \text{for } U \text{ aff sch (resp. alg space)} \\ G \text{ finite group}$$



$\exists V \rightarrow X$ finite étale where
 V is aff sch (resp. sch, alg space)

(Consider $V \times_x V \times_x \dots \times_x V \setminus \Delta \rightarrow X$)

Using this fact:

Prop: $R \rightrightarrows U$ finite ét. equiv relation of affine schemes

$\Rightarrow U/R$ is affine scheme.

Pf: $U/R \cong V/G$ for G free action of finite gp

on V affine which is affine by cor. again.

Also get some results on lifting k-pts for general
alg stacks:

Prop: X alg stack, $x \rightarrow \text{spec } k$ sep & qcqp,
 $x \in X(k)$. Then $\exists U \rightarrow X$ smooth with
 $u \in U(k)$ over x , U affine.

Pf: Take $V \rightarrow X$ smooth map from affine
 scheme, image containing x .

$$\begin{array}{ccc} V_x & \longrightarrow & V \\ \text{smooth} \downarrow & & \downarrow \text{smooth} \\ \text{Spec } k & \longrightarrow & X \end{array}$$

Let $u \in U_x$ be a closed point;
 then $k \rightarrow k(u)$ is a finite field extension,
 sep of degree d .

If $V' = V \times_{x_1} \dots \times_{x_n} V \setminus \Delta,$

then

$$\begin{array}{ccc} \text{Spec } k(u) & \longrightarrow & V_x \longrightarrow V \\ \downarrow & & \\ \text{Spec } k & \xrightarrow{\quad} & \\ & \updownarrow & \\ & \text{Spec } k & \longrightarrow [V'/S_d] \\ & \searrow & \downarrow s \\ & & X \end{array}$$

Choose a faithful representation

$$S_d \subseteq GL_n.$$

Including presentations (see 3.4.2 for details)

$$\Rightarrow [V'/S_d] \cong [V' \times^{S_d} GL_n / GL_n]$$

$$\text{Let } U = V' \times^{S_d} GL_n.$$

The point $\text{Spec } k \rightarrow [U/GL_n]$

$$\begin{array}{c} \updownarrow \\ P \rightarrow U \end{array}$$

$$\text{GL}_n\text{-torsor} \downarrow \\ \text{Spec } k$$

$$\Rightarrow P \cong \text{trivial GL}_n\text{-torsor} \\ (\text{since GL}_n\text{ torsors} \Leftrightarrow \text{loc. free sheaves!})$$

$$\Rightarrow \text{Jk-point} \quad \begin{array}{c} \exists \\ P \rightarrow \text{Spec } k \end{array}$$

Then $U \rightarrow [U/\text{GL}_n] \rightarrow X$ is smooth,

and $\text{Spec } k \rightarrow P \rightarrow U$ is a lift of

$$\text{Spec } k \rightarrow X.$$

How can we get U affine?

By above, may assume X is an algebraic space
and take a map $V \xrightarrow{\text{ét}} X$ w/image containing x .

Now Prop 4.4.8 $\Rightarrow$ in this situation the map

$V \rightarrow X$ is quasicompact since image is too
(look @ graph factorization).

Repeating the argument from above get that
the action of S_2 is free on $V' = V \times_{X_1} \dots \times_{X_n} V \setminus \Delta$

$\Rightarrow V'/S_2$ is quasicompact, & take an appropriate
affine open containing lifted k-pt.

